ON STABILITY OF MOTION OF NONHOLONOMIC SYSTEMS

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In his investigations on stability of motion Liapunov [1] proceeded with Lagrange's equations in independent determining coordinates which describe the motion of holonomic systems. It is known that Liapunov's theory of stability, developed through the efforts of many scientists, achieved great sucess [2]. At the same time the problem of stability of motion of nonholonomic systems has been developed insufficiently, even though starting with the work of Whittaker [3] and Bottema [4] a fairly large amount of literature is devoted to this subject; in particular the problem of stability of equilibrium under the action of potential forces is almost completely uninvestigated. In many papers on stabity of nonholonomic systems not only a unified approach to the problem is absent, but frequently also inconsistencies in the method of investigation and even in results are encountered (for more details see introduction to paper [5]. Yet, problems of stability for nonholonomic systems have the character of problems of conditional stability in the sense of Liapunov. This circumstance was apparently first noted by Chetaev (page 384, [2]).

In this paper, after presentation of the general formulation of the problem, the question of stability and instability of equilibrium in nonholonomic systems under the action of potential forces is studied. In particular the conditions for applicability of Lagrange's theorem on stability of equilibrium are elucidated. The effect of dissipative forces on stability of equilibrium in nonholonomic systems is also examined. In conclusion two illustrative examples are presented.

1. Let us examine a system of material points. Independent Lagrange coordinates of this system are designated by $q_1, \ldots, q_n$. Let the system be constrained by m ideal nonintegrable linear constraints of the form

$$\dot q_r = \sum_{i=1}^{k} b_{ri}(q_1, \ldots, q_n, t)\, \dot q_i + \beta_r(q_1, \ldots, q_n, t) \qquad (r = k+1, \ldots, n;\ k = n - m) \tag{1.1}$$

Possible displacements of points of the system are determined by variations δq_j of Lagrange coordinates q_j, connected by the following relationships

$$\delta q_r = \sum_{i=1}^{k} b_{ri}\delta q_i \qquad (r = k+1, \ldots, n) \tag{1.2}$$

Equations of motions of nonholonomic systems are obtained in various forms. For the sake of definiteness (but without limitations of generality) we shall examine the equation in the form of Appel

$$\frac{\partial S}{\partial \ddot q_i} = Q_i^*, \qquad Q_i^* = Q_i + \sum_{r=k+1} Q_r \qquad (i = 1, \ldots, k) \tag{1.3}$$

Here $S = S(q_1^{\cdot\cdot}, \ldots, q_k^{\cdot\cdot}; q_1^{\cdot}, \ldots, q_k^{\cdot}; q_1, \ldots, q_n, t)$ is the energy of acceleration, $Q_i{}^*$ are generalized forces corresponding to coordinates q_i, the variations of which are arbitrary; in this connection $Q_j (j=1, \ldots, n)$ designate generalized Lagrange forces corresponding to coordinates q_j.

Eqs. (1.3), together with equations of constraints (1.1), represent a simultaneous system of $k+m=n$ equations with an equal number of unknowns $q_j (j=1, \ldots, n)$.

We note that regardless of the form in which the equations of motion of nonholonomic systems are taken, in the form of (1.3) or some other form, for obtaining a closed system of equations it is necessary to add equations of kinematic constraints to equations of motion. This is one of characteristic differences of nonholonomic systems from holonomic systems with independent coordinates which determine the specifics of formulation of the problem on stability of motion.

Let us assume that the equations of motion of the nonholonomic system have some particular solution

$$q_j = f_i(t), \qquad \dot{q}_j = f_j^{\cdot}(t) \tag{1.4}$$

which satisfies initial conditions

$$q_{j0} = f_j(t_0), \qquad q_{j0}^{\cdot} = f_j^{\cdot}(t_0) \tag{1.5}$$

We shall compare unperturbed motion (1.4) with perturbed motions of the system which are possible for the same forces and constraints but for different initial conditions

$$q_{j0} = f_j(t_0) + \varepsilon_j, \qquad q_{j0}^{\cdot} = f_j^{\cdot}(t_0) + \varepsilon_j{}' \tag{1.6}$$

where perturbations ε_j and $\varepsilon_j{}'$ are some real constants sufficiently small in absolute value. However, in constrast to the case of a holonomic system, perturbations ε_j and $\varepsilon_j{}'$ cannot be taken as arbitrary but must satisfy certain conditions arising from conditions due to the nonholonomic character. In fact, substituting (1.6) into (1.1) we shall have

$$f_r^{\cdot}(t_0) + \varepsilon_r{}' = \sum_{i=1}^{k} b_{ri}(f_s(t_0) + \varepsilon_s; t_0)[f_i^{\cdot}(t_0) + \varepsilon_i{}'] + b_r(f_s(t_0) + \varepsilon_s; t_0)$$

Assuming functions $b_{ri}(q_1, \ldots, q_n, t_0)$ and $b_r(q_1, \ldots, q_n, t_0)$ to be holomorphic functions of q_j and expanding them in Taylor series, we obtain

$$\varepsilon_r{}' = \sum_{i=1}^{k} b_{ri}(f_s(t_0), t_0)\,\varepsilon_i{}' + \sum_{i=1}^{k}\sum_{j=1}^{n}\left(\frac{\partial b_{ri}}{\partial q_j}\right)_0 f_i^{\cdot}(t_0)\,\varepsilon_j + \sum_{j=1}^{n}\left(\frac{\partial b_r}{\partial q_j}\right)_0 \varepsilon_j + \cdots \tag{1.7}$$

which connect ε_j and $\varepsilon_j{}'$; dots designate members higher than first order of smallness with respect to perturbations.

The problem of stability of motion in the sense of Liapunov for nonholonomic systems can apparently be formulated in the same manner as for holonomic systems [1 and 2] under the condition that perturbations ε_j and $\varepsilon_j{}'$ satisfy conditions (1.7). Consequently the problem of stability of motion of a nonholonomic system has the character of the problem on conditional stability [2].

If we accept for nonholonomic systems the determination of conditional stability given by Liapunov, we can for the solution of problems on stability of motion of nonholonomic systems in this manner utilize methods which were worked out in the theory of stability of holonomic systems. In this case proofs of general theorems on stability and instability which form the basis of the first and second method of Liapunov remain the same for nonholonomic systems as they are for for holonomic systems.

2. Let us examine the problem of stability of equilibrium of a system constrained by stationary nonholonomic constraints

$$q_r^{\cdot} = \sum_{i=1}^{k} b_{ri}(q_1, \ldots, q_n)\, q_i^{\cdot} \qquad (r = k+1, \ldots, n) \tag{2.1}$$

and under the action of potential active forces derived from the force function $U(q_1, \ldots, q_n)$.

In accordance with the principle of virtual displacements it is necessary and sufficient for the equilibrium of the system that the force function U has a stationary value

$$\delta U = 0 \tag{2.2}$$

i.e. in the set of possible displacements δq_i the force function in the equilibrium position has a relative local extremum. For holonomic systems the character of this extremum determines, as is well known [2], the stability or instability of the equilibrium. We shall clarify the situation in the case of nonholonomic systems.

By virtue of (1.2) condition (2.2) is equivalent to the following Eqs. of equilibrium:

$$Q_i^* = \frac{\partial U}{\partial q_i} + \sum_{r=k+1}^{n} b_{ri} \frac{\partial U}{\partial q_r} = 0 \qquad (i = 1, \ldots, k) \tag{2.3}$$

These equations can of course also be obtained from Eqs. of motion (1.3). Since the number n of unknowns q_j in Eqs. (2.3) exceeds the number k of equations by the number m of nonholonomic constraints, the problem of searching for the equilibrium position of the nonholonomic system is generally speaking undetermined [5], locations of equilibrium form manifolds of dimension no less than m. Let us examine some point of the manifold of equilibrium positions and without loss of generality let us assume that for this point

$$q_j = 0, \quad q_j^{\cdot} = 0 \qquad (j = 1, \ldots, n) \tag{2.4}$$

Further we shall assume that the force function $U(q_1, \ldots, q_n)$ represents a holomorphic function of variables q_j. In the general case in the vicinity of point (2.4) it has the form

$$U = \sum_{j=1}^{n} a_j q_j + \frac{1}{2} \sum_{ij=1}^{n} c_{ij} q_i q_j + u(q_1, \ldots, q_n) \tag{2.5}$$

Here a_{ij}, $c_{ij} = c_{ji}$ are constants, while $u(q_1, \ldots, q_n)$ designates the total of terms of higher than second order of smallness. Taking into account (2.5) Eqs. of equilibrium (2.3) assume the form

$$Q_i^* = a_i + \sum_{j=1}^{n} c_{ij} q_j + \frac{\partial u}{\partial q_i} + \sum_{r=k+1}^{n} \left(a_r + \sum_{j=1}^{n} c_{rj} q_j + \frac{\partial u}{\partial q_r} \right) b_{ri} = 0 \tag{2.6}$$

Since it is assumed that point (2.4) belongs to a manifold of equilibrium positions, the following conditions must be satisfied

$$a_i + \sum_{r=k+1}^{n} a_r b_{ri}^{\circ} = 0 \quad (i = 1, \ldots, k), \qquad b_{ri}^{\circ} = b_{ri}(0, \ldots, 0) \tag{2.7}$$

It is apparent that if $a_r = 0$ or $b_{ri}^{\circ} = 0$ $(i = 1, \ldots, k;\ r = k+1, \ldots, n)$, then also all $a_i = 0$. If the latter condition is not fulfilled, this can be achieved by substitution of variables

$$u_r = q_r - \sum_{i=1}^{k} b_{ri}^{\circ} q_i \qquad (r = k+1, \ldots, n) \tag{2.8}$$

In this case if variables u_r are again designated through q_r, the force function will have the form (2. 5) where now all $a_i = 0$ $(i = 1,\dots, k)$. The equations of constraints will have the form (2. 1), where [6] all $b_{ri}^\circ = 0$ $(r = k+1,\dots, n)$. In the following, if for initial variables $b_{ri}^\circ \neq 0$, we shall assume that the substitution of variables (2. 8) has been performed

Note. In the fulfillment of conditions (2. 7) the nonholonomic system may be in equilibrium under the action of forces derived from function U which contains terms linear with respect to q_j. For a holonomic system with independent coordinates q_i such a case of equilibrium is impossible.

Let us assume that the functional determinant of the system of Eqs. (2. 6) with respect to variables $q_i (i = 1, \dots, k)$ for zero values of variables $q_j (j = 1, \dots, n)$ has the form

$$\Phi \equiv \frac{\partial (Q_1^*, \dots, Q_k^*)}{\partial (q_1, \dots, q_k)} = \left\| c_{ij} + \sum_r a_r \overset{\circ}{b}_{rij} \right\| \neq 0 \tag{2.9}$$

Here, for brevity, the following notations are introduced

$$b_{rij} = (\partial b_{ri} / \partial q_j)_0 \qquad (i = 1, \dots, k;\ j = 1, \dots, n;\ r = k+1, \dots, n)$$

Then the following solution for Eqs. (2. 6) exists

$$q_i = \varphi_i (q_{k+1}, \dots, q_n) \qquad (i = 1, \dots, k) \tag{2.10}$$

where φ_i are some holomorphic functions of q_r which disappear when all $q_r = 0$ $(r = k+1, \dots, n)$. Since Eqs. (2. 1) for $q_i = 0$ $(i = 1, \dots, k)$ have the solution

$$q_r = c_r \qquad (r = k+1, \dots, n) \tag{2.11}$$

where c_r are arbitrary constants then, apparently, equilibrium (2. 4) will belong to m parametric family of solutions (2. 10) and (2. 11) of equations of motion.

Solution (2. 4) will be taken as the unperturbed solution and its stability will be examined; equations of perturbed flow will have the form (1. 3), (2. 1). Let us examine their structure. For this purpose instead of using Eqs. (1. 3) it is more convenient to make use of equivalent equations in the form of Voronets [7].

$$\frac{d}{dt}\frac{\partial \Theta}{\partial q_i^{\cdot}} - \frac{\partial (\Theta + U)}{\partial q_i} - \sum_{r=k+1}^{n} \frac{\partial (\Theta + U)}{\partial q_r} b_{ri} = \sum_{r=k+1}^{n} \theta_r \sum_{j=1}^{k} A_{ij}^{(r)} q_j^{\cdot} \qquad (i = 1, \dots, k) \tag{2.12}$$

Here

$$\Theta(q_1, \dots, q_n;\ q_1^{\cdot}, \dots, q_k^{\cdot}) = \frac{1}{2} \sum_{ij=1}^{k} a_{ij} (q_1, \dots, q_n)\, q_i^{\cdot} q_j^{\cdot}$$

denotes the kinetic energy of the system T expressed with the aid of Eqs. of constraints (2. 1) only through independent velocities $q_i^{\cdot} (i = 1, \dots, k)$, through which generalized impulses θ_r, corresponding to dependent velocities q_r, are also expressed

$$\theta_r (q_1, \dots, q_n, q_1^{\cdot}, \dots, q_k^{\cdot}) = \frac{\partial T}{\partial q_r^{\cdot}} \qquad (r = k+1, \dots, n)$$

Apparently θ_r are linear homogeneous forms of velocities q_i in the examined case with stationary constraints (2. 1). The coefficients $A_{ij}^{(r)}$ are expressed in the following manner through the coefficients b_{ri} of the equations of constraints

$$A_{ij}^{(r)} = \frac{\partial b_{ri}}{\partial q_j} + \sum_{s=k+1}^{n} \frac{\partial b_{ri}}{\partial q_s} b_{sj} - \frac{\partial b_{rj}}{\partial q_i} - \sum_{s=k+1}^{n} \frac{\partial b_{rj}}{\partial q_s} b_{si}$$

Since the coefficients $A_{ij}^{(r)}$ are antisymmetric with respect to indices i and j

$$A_{ij}^{(r)} = - A_{ji}^{(r)}$$

the following identity is applicable

$$\sum_{r=k+1}^{n} \theta_r \sum_{i,\,j=1}^{k} A_{ij}^{(r)} q_i^{\cdot} q_j^{\cdot} = 0$$

i. e. terms appearing in the right-hand parts of Eqs. (2. 12) have gyroscopic structure. If Eqs. of constraints (2. 1) are integrable then all

$$A_{ij}^{(r)} = 0 \qquad (i,\, j = 1, \ldots, k;\; r = k+1, \ldots, n)$$

and Eqs. (2. 12) transform into Lagrange equations in redundant coordinates. In this manner terms of nonholonomic character in the right-hand side of Eqs. (2. 12) are equivalent to gyroscopic forces. It is only necessary to keep in mind that these forces in this case are quadratic with respect to velocities q_i, and if equations of perturbed motion are linearized, they will not appear in the latter, i. e. in the first approximation they have no effect on the motion of the system. In the particular case of Chaplygin's systems [8], if coordinates q_r corresponding to eliminated velocities do not enter explicitly into expressions of force function U, of coefficients of kinetic energy T and of coefficients b_{ri} of constraint equations, then Eqs. (2. 12) take the form of Chaplygin's equations

$$\frac{d}{dt}\frac{\partial\Theta}{\partial q_i^{\cdot}} - \frac{\partial(\Theta + U)}{\partial q_i} = \sum_{r=k+1}^{n} \theta_r \sum_{j=1}^{k} A_{ij}^{(r)} q_j^{\cdot} \qquad (2.13)$$

Since these equations can be integrated independently of Eqs. of constraints (2. 1), for Chaplygin's systems the problem of stability of motion with respect to certain functions q_i, $q_i^{\cdot}$, t $(i = 1, \ldots, k)$ can be formulated as problem of unconditional stability in the sense of Liapunov.

It is not difficult to see that for stationary constraints and potential forces, Eqs. of perturbed motion (1. 3) or (2. 12) and (2. 1) assume the energy integral as

$$H = T - U = \text{const} \quad \text{or} \quad H = \Theta - U = \text{const} \qquad (2.14)$$

The kinetic energy of system T (or Θ) is a positive definite quadratic form in the velocities $\dot{q}_j$ (or $\dot{q}_i$). We note that by virtue of existence of integral (2. 14) the position of equilibrium of a nonholonomic system cannot be asymptotically stable under the influence of potential forces alone.

Under certain conditions Lagrange's theorem of stability of equilibrium is applicable to nonholonomic systems. In fact, for point (2. 4) let the following conditions be satisfied

$$\partial U \,/\, \partial q_j = 0 \qquad (j = 1, \ldots, n) \qquad (2.15)$$

indicating that the position of equilibrium of the system is a stationary point of the function U $(q_1, \ldots, q_n)$. In this connection the function U apparently does not contain terms linear in q_r, i. e. all $a_r = 0$. Under these conditions on the basis of Liapunov's theorem of stability or on the basis of the theorem of stability with respect to a part of variables [9], taking as Liapunov's function the energy of the system H we become convinced that for nonholonomic systems Lagrange's theorem is applicable.

Theorem 2. 1. If in the vicinity of the location of equilibrium of a nonholonomic system the force function U $(q_1, \ldots, q_n)$ is negative definite with respect to variables q_s $(s = 1, \ldots, p \leqslant n)$, then the position of equilibrium is stable with respect to q_s and $\dot{q}_j$ $(j = 1, \ldots, n)$.

In essence this theorem is obvious. In fact, if nonintegrable constraints coincident with the position of equilibrium are placed on a holonomic system in a stable state of equilibrium, the stability is not violated. However a nonholonomic system can be in stable equilibrium even in cases when the force function in the equilibrium position does not have a maximum with respect to coordinates q_j, in particular when it contains linear terms. Let us see what conclusion about stability of equilibrium can be extracted in this case from the sign of second variation of force function U.

Utilizing Eqs. (1.2) and taking into account that $b_{ri}^{\circ} = 0$, we find for point (2.4)

$$\delta^2 U = \frac{1}{2} \sum_{ij=1}^{k} c_{ij} \delta q_i \delta q_j \tag{2.16}$$

Let us examine function $V = \Theta^* - U^*$ where Θ^* and U^* designate function Θ and U if in the latter all variables $q_r = 0$ $(r = k+1, \ldots, n)$,

$$\Theta^* (q_1, \ldots, q_k, q_1^{\cdot}, \ldots, q_k^{\cdot}) = \Theta (q_1, \ldots, q_k, 0, \ldots, 0, q_1^{\cdot} \ldots q_k^{\cdot}),$$

$$U^* (q_1 \ldots, q_k) = U (q_1 \ldots q_k, 0, \ldots 0)$$

Expansion of function U^* in Maclaurin's series in the vicinity of point (2.4) starts with a quadratic form, where

$$\delta^2 U^* = \delta^2 U$$

By virtue of Eqs. of perturbed motion (2.12) in which we set $q_r = 0$, the time derivative of function V is

$$V^{\cdot} = \sum_{i=1}^{k} q_i^{\cdot} \sum_{r=k+1}^{n} \left[\frac{\partial (\Theta + U)}{\partial q_r} b_{ri} \right]_{q_r = 0} \tag{2.17}$$

This expression is equal to zero with accuracy to terms of not lower than third order of smallness with respect to q_i and $\dot{q}_i$, at least in cases when the force function U does not contain linear terms with respect to q_r

$$a_r = \left(\frac{\partial U}{\partial q_r} \right)_0 = 0 \qquad (r = k+1, \ldots, n) \tag{2.18}$$

or when coefficients b_{ri} of Eqs. of constraints (2.1) satisfy conditions

$$\overset{\circ}{b}_{rij} \equiv \left(\frac{\partial b_{ri}}{\partial q_j} \right)_0 = 0 \qquad (i = 1, \ldots, k;\ j = 1, \ldots, k;\ r = k+1, \ldots, n) \tag{2.19}$$

Taking V as Liapunov's function, we arrive on the basis of Liapunov's theorem on stability at the conclusion that the following theorem is valid.

Theorem 2.2. If in equilibrium position (2.4) of a nonholonomic system conditions (2.18) or (2.19) are satisfied and the second variation $\delta^2 U$ of the force function U is negative definite, the position of equilibrium is stable in the first approximation with respect to q_i and $\dot{q}_j$.

Corollary. When conditions (2.18) or (2.19) are fulfilled the nonholonomic character of the system does not have essential significance for small oscillations near the equilibrium position.

Note. A statement about the insignificance of nonholonomic character for small oscillations near the position of equilibrium was shown by Whittaker [3] without any conditions, even though, from his development it might be seen that he implicity assumed the fulfillment of conditions of the form (2.18). In the general case, as was first noted by Bottema [4], this statement is not correct. Here the corresponding characteristic determant will be unsymetrical in contrast to the case of a holonomic system.

We shall now turn to the examination of the question of instability of equilibrium of a nonholonomic system under the action of potential forces. Let, in an arbitrarily small region in the vicinity of equilibrium position (2.4), the force function U be able to assume positive values and in the equilibrium position $U=0$. In this connection it is assumed that for a given value of the largest permissible deviation A, positions arbitrarily close to the unperturbed position, for which $U>0$, are possible with consideration of constraints (2.1) placed on the system.

Let us examine the function

$$W = -H \sum_{i=1}^{k} \frac{\partial \Theta}{\partial q_i^{\cdot}} q_i$$

In the region of small values in absolute magnitude of coordinates q_j and velocities $q_j^{\cdot}$ we select a region C which exists under our assumptions for arbitrarily small, in absolute magnitude, values of q_j and $q_j^{\cdot}$. This region C is defined by simultaneous inequalities

$$H<0, \qquad \sum_{i=1}^{k} \frac{\partial \Theta}{\partial q_i^{\cdot}} q_i > 0 \tag{2.20}$$

The total derivative with respect to time of the function W has, by virtue of Eqs. of perturbed motion (2.12), the form

$$W^{\cdot} = -H \sum_{i=1}^{k} \left(q_i^{\cdot} \frac{\partial \Theta}{\partial q_i^{\cdot}} + q_i \frac{d}{dt} \frac{\partial \Theta}{\partial q_i^{\cdot}} \right) = -H \Big[2\Theta + \sum_{i=1}^{k} q_i \Big(\frac{\partial \Theta}{\partial q_i} + \sum_{r=k+1}^{n} \frac{\partial \Theta}{\partial q_r} b_{ri} \Big) +$$
$$+ \sum_{i,j=1}^{k} \sum_{r=k+1}^{n} A_{ij}^{(r)} \Theta_r q_i q_j + \sum_{i=1}^{k} q_i \Big(\frac{\partial U}{\partial q_i} + \sum_{r=k+1}^{n} \frac{\partial U}{\partial q_r} b_{ri} \Big) \Big]$$

Since constraints placed on the system are assumed to be independent of time, the kinetic energy of system Θ represents a positive definite function of $q_i^{\cdot}$. For values of coordinates q_j, sufficiently small in absolute magnitude, the function

$$2\Theta + \sum_{i=1}^{k} q_i \Big(\frac{\partial \Theta}{\partial q_i} + \sum_{r=k+1}^{n} \frac{\partial \Theta}{\partial q_r} b_{ri} \Big) + \sum_{i,j=1}^{k} \sum_{r=k+1}^{n} A_{ij}^{(r)} \Theta_r q_i q_j^{\cdot}$$

will also be positive definite with respect to velocities $q_i^{\cdot}$. Then, if in region C Expression

$$\sum_{i=1}^{k} q_i \Big(\frac{\partial U}{\partial q_i} + \sum_{r=k+1}^{n} \frac{\partial U}{\partial q_r} b_{ri} \Big) \tag{2.21}$$

will be a positive definite function of q_i, the function $W^{\cdot}$ will be a positive definite function of $q_i^{\cdot}$ and q_i in the region C. In this connection all conditions of Chetaev's theorem on instability will be fulfilled. On this basis we conclude that the following theorem is valid.

T h e o r e m 2. 3. If in an arbitrarily small region in the vicinity of the position of equilibrium of a nonholonomic system the force function U can assume positive values, whereby Expression (2.21) in region (2.20) is a positive definite function of q_i, the position of equilibrium is unstable with respect to $q_i^{\cdot}$ and q_i.

Corollary. If the force function does not depend on coordinates q_r, the position of equilibrium (2.4) of the nonholonomic system is unstable when [2]:

a) the force function U represents some homogeneous form U_p of degree p in variables q_i $(i = 1,\ldots, k)$ and for arbitrarily small, in absolute magnitude, values of variables q_i it can assume positive values or

b) the force function has the form $U(q_1,\ldots, q_k) = U_p + U_{p+1} + \ldots$ and for arbitrarily small q_i $(i = 1,\ldots, k)$ it can assume positive values. Here the signs of expressions U and $pU_p + (p+1)U_p + \ldots$ are determined by the form of U_p. We proceed to the examination of the case when in an arbitrarily small region in the vicinity of the equilibrium position the function U^* can assume positive values. Let us examine the function

$$W = -(\Theta^* - U^*)\sum_{i=1}^{k}\frac{\partial\Theta^*}{\partial q_i^{\cdot}}q_i$$

and the region of arbitrarily small, in absolute magnitude, values q_i and $q_i^{\cdot}$. This region is defined by the simultaneous inequalities

$$\Theta^* - U^* < 0, \qquad \sum_{i=1}^{k}\frac{\partial\Theta^*}{\partial q_i^{\cdot}}q_i > 0 \tag{2.22}$$

By virtue of Eqs. (2.12) of perturbed flow in which we take $q_r = 0$, the total derivative of the function W with respect to time is

$$W^{\cdot} = -(\Theta^* - U^*)\Big\{2\Theta + \sum_{i=1}^{k} q_i\Big(\frac{\partial\Theta}{\partial q_i} + \sum_{r=k+1}^{n}\frac{\partial\Theta}{\partial q_r}b_{ri}\Big) + \sum_r \Theta_r \sum_{i,j=1}^{k} A_{ij}^{(r)} q_i q_j^{\cdot} +$$
$$+ \sum_{i=1}^{k}\Big(\frac{\partial U}{\partial q_i} + \sum_{r=k+1}^{n}\frac{\partial U}{\partial q_r}b_{ri}\Big)q_i\Big\}_{q_r=0} - \sum_{i,j=1}^{k} q_i^{\cdot} q_j \frac{\partial\Theta^*}{\partial q_j^{\cdot}}\sum_{r=k+1}^{n}\Big[\frac{\partial(\Theta+U)}{\partial q_r}b_{ri}\Big]_{q_r=0}$$

On the basis of Chetaev's theorem on instability we arrive at the conclusion that the following theorem is valid.

Theorem 2.4. If the second variation $\delta^2 U$ of the force function can assume positive values and conditions (2.18) or (2.19) are satisfied, the position of equilibrium is unstable.

3. Let us examine the effect of dissipative forces on stability of equilibrium position of a nonholonomic system. In addition to potential forces let dissipative forces also act on the system

$$Q_i^{\circ} = -\partial f / \partial q_i^{\cdot} \qquad (i = 1, \ldots, k) \tag{3.1}$$

These forces are derived from a positive definite Rayleigh function

$$2f = \sum_{i,j=1}^{k}\alpha_{ij}q_i^{\cdot}q_j^{\cdot}$$

In this case the equations of perturbed motion of the system near the equilibrium position differ from Eqs. (2.12) only by the addition of terms (3.1) in the right-hand parts of the latter.

From equations of perturbed flow the following Eq.

$$\frac{d}{dt}(\Theta - U) = -2f \tag{3.2}$$

follows for the rate of energy dissipation of the system.

Theorem 3.1. When conditions of Theorem 2.1 or 2.2 are satisfied, dissipative forces do not violate the stability of the equilibrium position of the system.

Proof. If conditions of Theorems 2.1 or 2.2 are satisfied, then the validity of this statement follows from Liapunov's theorem on stability, if in the first case $H=\Theta-U$ is taken as Liapunov's function and in the second case $V=\Theta^*-U^*$. Here, instead of Eq. (2.17) we will have

$$V^{\cdot} = -2f + \sum_{i=1}^{k} \sum_{r=k+1}^{n} \left[\frac{\partial(\Theta+U)}{\partial q_r} b_{ri}\right]_{q_r=0} q_i^{\cdot} \tag{3.3}$$

Theorem 3.2. If in the expansion of force function U there are no linear terms and if in an arbitrarily small region in the vicinity of the equilibrium position of a non-holonomic system, the function U and Expression (2.21) are negative definite with respect to variables q_s $(s = 1,\ldots, k)$, the equilibrium becomes asymptotically stable with respect to variables q_s and $q_j^{\cdot}$ on addition of dissipative forces.

Proof. Let us examine the function

$$V = H + \beta \sum_{i=1}^{k} q_i \frac{\partial\Theta}{\partial q_i^{\cdot}} \tag{3.4}$$

The positive constant β can always be selected so small that the function V will be positive definite. By virtue of equations of perturbed motion the total derivative with respect to time of function V is

$$V^{\cdot} = \beta\left[-2\left(\frac{f}{\beta} - \Theta\right) - \sum_{i=1}^{k} q_i \frac{\partial f}{\partial q_i^{\cdot}} + \sum_{i=1}^{k} q_i \left(\frac{\partial U}{\partial q_i} + \sum_{r=k+1}^{n} \frac{\partial U}{\partial q_r} b_{ri}\right) + \ldots\right] \tag{3.5}$$

where dots designate terms of no less than third order of smallness with respect to q_i and $q_i^{\cdot}$. For sufficiently small positive β the function $V^{\cdot}$ will be negative definite ([2], p. 77). Consequently, all conditions of Liapunov's theorem on asymptotic stability are satisfied, which proves the Theorem.

The proof of this theorem can be presented in a different way. According to Eq. (3.2) the total mechanical energy of the system in its perturbed motion is dissipated until all $q_i^{\cdot}$ $(i = 1, \ldots, k)$ becomes equal to zero. However, under the conditions of the Theorem this is possible only at the point $q_i = 0$ $(i = 1,\ldots, k)$, since from the property of Expression (2.21) of having a fixed sign it follows that in the vicinity of the equilibrium position the generalized forces $Q_i^* \neq 0$, as long as $q_i \neq 0$ $(i = 1, \ldots, k)$.

Corollary. If the force function does not depend on coordinates q_r, Expression (2.21) takes the form

$$\sum_{i=1}^{k} q_i \frac{\partial U}{\partial q_i} = 2U_2 + \ldots \tag{3.6}$$

and if U_2 is a negative definite quadratic form of q_1, the equilibrium position becomes asymptotically stable on addition of dissipative forces.

Theorem 3.3. If in an arbitrarily small region in the vicinity of the equilibrium position the quadratic part of function U^* is negative definite and conditions (2.18) or (2.19) are satisfied, on addition of dissipative forces the position of equilibrium becomes asymptotically stable in the first approximation with respect to variables q_i and $q_j^{\cdot}$.

Proof. Let us examine the positive definite function

$$V = \Theta^* - U^* + \beta \sum_{i=1}^{k} q_i \frac{\partial \Theta^*}{\partial q_i^{\cdot}} \tag{3.7}$$

and its total derivative with respect to time.

By virtue of equations of perturbed motion in which we shall take $q_r = 0$

$$V^{\cdot} = -2(f - \beta\Theta^*) - \beta \sum_{i=1}^{k} q_i \frac{\partial f}{\partial q_i^{\cdot}} + \beta \sum_{i=1}^{k} q_i \left(\frac{\partial U}{\partial q_i} + \sum_{r=k+1}^{n} \frac{\partial U}{\partial q_r} b_{ri} \right)_{q_r=0} + \ldots \tag{3.8}$$

Dots designate terms of no less than third order of smallness with respect q_1 and $q_i^{\cdot}$.

For conditions indicated, all conditions of Liapunov's theorem on asymptotic stability are satisfied, which proves our statement.

Theorem 3.4. The equilibrium position of a nonholonomic system which is unstable when conditions of Theorems 2.3 or 2.4 are fulfilled, cannot be stabilized through dissipative forces.

Proof. Let conditions of Theorem 2.3 be satisfied. Then in a region of arbitrarily small values, in absolute magnitude of q_1 and $q_i^{\cdot}$, we can select a region defined by inequalities (2.20). Initial perturbations are selected in this region. The system left to itself will move then according to Eq. (3.2). It follows from this that the total mechanical energy of the system, being negative at the initial moment of time, will decrease until all $q_i^{\cdot}$ $(i = 1, \ldots, n)$ will become zero. However, in region (2.20), Expression (2.21) has a fixed sign, as a result of which generalized forces Q_i^* do not become zero at any point of region (2.20) with the exception of the origin of coordinates $q_1 = 0$. Then on the basis of inequality

$$\Theta - U \leqslant \Theta_0 - U_0 < 0 \tag{3.9}$$

we arrive at the conclusion that a nonholonomic system eventually will leave any arbitrarily small region in the vicinity of equilibrium position (2.4). The proof of instability in case of fulfillment of conditions of Theorem 2.4 is analogous.

4. Examples. 1. Let us examine a heavy homogeneous body of revolution with a spherical base resting on a horizontal absolutely rough surface. The center of gravity O of the body is taken as the origin of the system of coordinates $Oxyz$. This system of coordinates is rigidly connected with the body. The axis z of this system is oriented upward along the axis of revolution of the body.

The coordinate of the geometrical center O_1 of the spherical base is designated by a_1 on this axis. The radius of the base is designated by a. The horizontal plane will be taken as the plane $\xi\eta$ in the stationary system of coordinates $\xi\eta\zeta$ with the axis ζ pointing vertically upward.

The position of the body will be determined by coordinates ξ and η of its point of contact with the plane and Euler's angles θ, ψ and φ. The potential energy of the body is

$$V = Mg\,(a - a_1 \cos\theta)$$

where M is the mass of the body, g is the gravitational acceleration. The position of equilibrium of the body on the surface are determined from Eq.

$$\partial V / \partial\theta = Mga_1 \sin\theta = 0 \cdot$$

In the position of equilibrium let $\theta = 0$.

Since

$$[\partial^2 V / \partial\theta^2]_{\theta=0} = Mga_1, \qquad \theta \frac{\partial V}{\partial\theta} = Mga_1 (\theta^2 + \ldots)$$

this position of equilibrium is, according to Theorem 2.1, stable with respect to $\xi^\cdot$, $\eta^\cdot$, $\theta^\cdot$, $\varphi^\cdot$, $\psi^\cdot$ and θ, if the center of gravity of the body is located below point O_1 ($a_1 > 0$) and unstable according to Theorem 2.3 if the center of gravity is located above point O_1 ($a_1 < 0$). The dissipative force $-a\theta^\cdot$ sets the stability according to Theorem 3.2 as far as asymptotic stability in the case $a_1 > 0$.

2. Let us examine the problem of Kerkhoven-Vithoff on stability of equilibrium of two heavy homogeneous bodies which have the shape of hemispheres. One of these, with radius R_1, rests with its spherical surface on the horizontal plane, and the other, with radius R_2, rests on the top flat base of the first body. The surfaces of contact are considered absolutely rough. Retaining the nomenclature ([3], p. 250) we take as Lagrange's coordinates of the system the quantities $\alpha_2, \beta_3, \gamma_1, a_2, b_3, c_1, \alpha$ and β, a, b.

Conditions of contact of the first body with the plane and the second body with the first have the form

$$c = R_1 - c_3 l_1, \qquad \gamma = R_2 + l_1 - l_2\gamma_3 \tag{4.1}$$

while conditions of nonholonomic character have the form

$$\begin{aligned} \alpha^\cdot &= R_2(\gamma_2 - \alpha_2\gamma_1)\alpha_2^\cdot - (R_2\gamma_1^2 + R_2\gamma_3 - l_2)\gamma_1^\cdot \\ \beta^\cdot &= -R_2(\gamma_1 + \alpha_2\gamma_2)\alpha_2^\cdot + [R_2(1 - {}^1\!/_2(\gamma_1^2 + \beta_3^2) - \beta_3\gamma_2) - l_2]\beta_3 \end{aligned} \tag{4.2}$$

In Lagrange's coordinates, with accuracy to a constant, the potential energy of the system has the form

$$\begin{aligned} V = {}^1\!/_2(M_1 + M_2)gl_1(c_1^2 + b_3^2) + M_2g\,[c_1\alpha - b_3\beta + \\ + {}^1\!/_2 l_2(\gamma_1^2 + \beta_3^2) - {}^1\!/_2(R_2 + l_1 - l_2)(c_1^2 + b_3^2)] + \ldots \end{aligned} \tag{4.3}$$

Eqs. of equilibrium (2.3)

$$\begin{gathered} M_2gR_2c_1(\gamma_2 - \alpha_2\gamma_1) + M_2gR_2b_3(\gamma_1 + \alpha_2\gamma_2) = 0 \\ M_2gl_2\beta_3 - M_2gb_3[R_2(1 - {}^1\!/_2(\gamma_1^2 + \beta_3^2) - \beta_3\gamma_2) - l_2] = 0 \\ M_2gl_2\gamma_1 - M_2gc_1(R_2\gamma_1^2 + R_2\gamma_3 - l_2) = 0 \\ [M_1l_1 - M_2\beta - M_2(R_2 - l_2)]gb_3 = 0 \\ M_2g\alpha + M_1gl_1c_1 - M_2g(R_2 - l_2)c_1 = 0 \end{gathered} \tag{4.4}$$

permit the solution

$$\beta_3 = \gamma_1 = b_3 = c_1 = \alpha = \beta = 0 \tag{4.5}$$

For position of equilibrium (4.5) function V does not have a minimum and conditions of Lagrange's theorem are not satisfied. Let us apply Theorem 2.2. Linearizing Eqs. of constraints (4.2) and integrating we will have

$$\alpha = -(R_2 - l_2)\gamma_1, \qquad \beta = (R_2 - l_2)\beta_3 \tag{4.6}$$

Substituting (4.6) into Expression (4.3) and taking into account that $l_i = {}^3\!/_8 R_i$ ($i = 1, 2$), we will obtain

$$\begin{aligned} V^* = g/16\,[(3M_1R_1 - 5M_2R_2)(c_1^2 + b_3^2) - 10M_2R_2(c_1\gamma_1 + b_3\beta_3) + \\ + 3M_2R_2(\gamma_1^2 + \beta_3^2)] + \ldots \end{aligned} \tag{4.7}$$

The function V^* will be a positive definite function of variables β_3, γ_1, b_3 and c_1 under the condition

$$9M_1R_1 > 40M_2R_2 \tag{4.8}$$

which according to Theorem 1.2 in the first approximation is the condition of stability (4.5) with respect to indicated variables and generalized velocities.

On addition of dissipative forces, derived from quadratic function of Rayleigh which is positive definite with respect to α_2', β_3', γ_1', a_2', b_3' and c_1', the position of equilibrium (4.5) becomes in this case asymptotically stable in the first approximation.

In the case when

$$9M_1R_1 < 40M_2R_2 \tag{4.9}$$

function V^* can acquire negative values. According to Theorem 2.3, the position of equilibrium (4.5) will be unstable.

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